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A note on positivity of Einstein bundles

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ABSTRACT

In this note we give a criterion for the positivity of the curvature tensor of a Hermitian Einstein metric in a holomorphic vector bundle. This is a differential geometric version of an algebraic ampleness criterion previously proved by M. Schneider and A. Tancredi.

1. INTRODUCTION

For holomorphic vector bundles on complex manifolds the algebraic notion of *ampleness* in the sense of Hartshorne [H] and the differential geometric notion of *positivity* in the sense of Griffiths [G] are strongly related; the latter always implies the first, and for line bundles they are equivalent. For bundles on surfaces, M. Schneider and A. Tancredi [ST] proved the following "numerical" ampleness criterion.

(1.1) THEOREM. *Let E be a holomorphic rank 2 vector bundle on a compact complex surface X with the following properties*:

a) *The first two Chern classes $c_1(E), c_2(E)$ satisfy the inequality $c_1^2(E) > 2c_2(E) > 0$ in $H^4(X,\mathbb{Z}) \cong \mathbb{Z}$.*

b) *The restriction $E|_C$ is ample for all integral curves $C \subset X$.* (*This together with condition* a) *implies that* $\det E$ *is ample.*)

c) *E is stable with respect to* $\det E$.

Then E is ample.

An example to which this applies [S] is the cotangent bundle of a *Kodaira fibration*, i.e. a surface X for which there is a regular holomorphic map

$X \xrightarrow{p} S$ onto a curve S s.t. p is not a fibre bundle (compare [BPV]).

In this note we give a criterion (Theorem 2.1) which guaranties, under differential geometric conditions almost corresponding to those above, positivity of a bundle over a Kähler manifold; the main idea (due to M. Schneider) is as follows:

Positivity of E means the existence of a hermitian metric in E with positive curvature tensor (for precise definitions see section 2); for the cotangent bundle of a Kodaira fibration such a metric has been constructed by I.-H. Tsai [T]. The stability condition in Theorem 1.1 is, by the solution of the Kobayashi-Hitchin conjecture (see [D1],[D2],[K],[L],[UY1],[UY2]), equivalent to the existence of an essentially unique metric which is (irreducibly) Hermitian-Einstein (*HE*), this also is defined in terms of the curvature. Hence one can ask whether this natural metric in fact has positive curvature; for this we translate the requirement on the Chern classes in Theorem 1.1 into a similar statement on the Chern forms given by the *HE*-metric, and using $\det E$ as the ample line bundle defining stability is equivalent to saying that the first Chern form of the *HE*-metric equals (up to a constant) the given Kähler form on X.

Theorem 2.1 requires only $\dim X = 2$ *or* $\operatorname{rk} E = 2$; in case of a 2-bundle on a surface it reads

(1.2) THEOREM. *Let (X, g) be a compact Kähler surface and (E, h, g) a Hermitian bundle on X such that the first Chern form $c_1(E, h)$ equals the Kähler form ω up to a positive constant. Suppose further that the first and second Chern form satisfy*

$$(*) \qquad c_1^2(E,h) - 2c_2(E,h) = \phi \cdot \omega^2$$

with a positive function ϕ. Then (E, h) is positive.

A Kodaira fibration X admits a Kähler Einstein metric g by Yau's solution of the Calabi conjecture; in particular is T^*X a *HE*-bundle such that $c_1 \triangleq \omega$, but unfortunately we cannot verify the crucial condition $(*)$ in this case. Also, from the construction of Tsai it does not follow that his metric is in fact Kähler Einstein, so the question remains if Einstein metrics are really good candidates for positive curvature (compare section 3).

Most of this work was done during two visits* at the University of Bayreuth in May 1989 and May 1990; the author would like to thank M. Schneider for the invitation.

2. THE POSITIVITY CRITERION

A general reference for the differential geometry used in the following is [K].

Let (X, g) be an n-dimensional compact complex Kähler manifold with Kähler form ω and $E \to X$ a holomorphic vector bundle of rank r. A hermitian metric h in E induces a unique h-unitary connection D_h in E which is compat-

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ible with the holomorphic structure; its curvature $F_h = D_h^2$ is a $(1,1)$-form with values in the endomorphisms of E.

DEFINITION. *We say that h has positive curvature if for every differentiable section s of E the $(1,1)$-form*

$$\Phi_s := \frac{\sqrt{-1}}{2} \cdot h(F_h(s), s)$$

is positive, i.e. locally one has $\Phi_s = \sqrt{-1}/2 \cdot \sum_{i,j=1}^n \phi_{ij} dz_i \wedge d\bar{z}_j$ where the matrix (ϕ_{ij}) is positive definite in all points which are no zeroes of s.

DEFINITION. *The metric h is called Hermitian Einstein (HE) if there is a constant $\lambda \in \mathbb{R}$ such that*

$$F_h \wedge \omega^{n-1} = -\sqrt{-1} \cdot \lambda \cdot id_E \otimes \omega^n.$$

A *HE*-metric, if it exists, is unique up to positive constants, and the factor λ depends only on the first Chern class of E.

The Chern forms $c_k(E, h)$ are the global closed forms of type (k, k) defined by

$$\det\left(id_E + \frac{\sqrt{-1}}{2\pi} \cdot F_h\right) = \sum_{k=1}^r c_k(E, h);$$

in particular one has

$$c_1(E, h) = \frac{\sqrt{-1}}{2\pi} \cdot \operatorname{tr} F_h.$$

Our main result is the following:

(2.1) THEOREM. *Let h be a Hermitian Einstein metric in E with factor $\lambda > 0$ such that $c_1(E, h) = r\lambda/2\pi \cdot \omega$. Suppose further that*

a) $\quad n = 2$ *and* $c_1^2(E, h) - \dfrac{2r(r-1)}{r^2 - 2r + 2} \cdot c_2(E, h) = \psi \cdot \omega^2$

or

b) $\quad r = 2$ *and* $\left(c_1^2(E, h) - \dfrac{4(n-1)^2}{n^2 - 2n + 2} \cdot c_2(E, h)\right) \wedge \omega^{n-2} = \psi \cdot \omega^n,$

where in both cases ψ is a strictly positive function. Then h has positive curvature.

REMARKS. i) If $n = r = 2$, then the coefficient of c_2 in both cases is 2.
ii) For a *HE*-metric h one always has $(c_1^2(E, h) - (2r/(r-1)) \cdot c_2(E, h)) \wedge \omega^{n-2} = \sigma \cdot \omega^n$ with a *nonpositive* function σ; this is not in conflict with our conditions because of $2r(r-1)/(r^2 - 2r + 2) < 2r/r - 1$ for all $r \geq 2$ and $4(n-1)^2/(n^2 - 2n + 2) < 4$ for all $n \geq 2$.

PROOF.

a) The curvature of a h-unitary connection is skew-adjoint, i.e. $F_h^* = -F_h$ where $*$ means adjoint with respect to h. The Hodge-$*$-operator associated to g induces a splitting

$$F_h = F_+ + F_-, \quad *F_\pm = \pm F_\pm;$$

one has

$$F_\pm^* = -F_\pm, \quad F_+ \wedge F_- = 0, \quad F_- \wedge \omega = 0.$$

From the HE-condition and the assumption on $c_1(E,h)$ we furthermore conclude

$$F_+ = -\sqrt{-1}\cdot\lambda\cdot id_E \otimes \omega, \quad \operatorname{tr} F_- = 0,$$

so

$$c_1^2(E,h) = -\frac{1}{4\pi^2}\cdot \operatorname{tr}(F_h)\wedge \operatorname{tr}(F_h) = \frac{r^2\cdot\lambda^2}{4\pi^2}\cdot\omega^2,$$

$$\begin{aligned} c_2(E,h) &= -\frac{1}{8\pi^2}\cdot \operatorname{tr}(F_h)\wedge\operatorname{tr}(F_h) + \frac{1}{8\pi^2}\cdot\operatorname{tr}(F_h\wedge F_h) \\ &= \frac{r^2\cdot\lambda^2}{8\pi^2}\cdot\omega^2 + \frac{1}{8\pi^2}\cdot\operatorname{tr}(F_+\wedge F_+) + \frac{1}{8\pi^2}\cdot\operatorname{tr}(F_-\wedge F_-) \\ &= \frac{r^2\cdot\lambda^2}{8\pi^2}\cdot\omega^2 + \frac{1}{8\pi^2}\cdot\operatorname{tr}(-\lambda^2\cdot id_E\otimes\omega^2) + \frac{1}{8\pi^2}\cdot\operatorname{tr}(F_-\wedge *F_-^*) \\ &= \frac{\omega^2}{8\pi^2}\cdot(r(r-1)\cdot\lambda^2 + \|F_-\|^2), \end{aligned}$$

where $\| \ \|$ is the pointwise Hodge norm induced by g and h; in particular it follows $c_2(E)>0$ (compare Theorem 1.1 a)). This implies

$$c_1^2(E,h) - \frac{2r(r-1)}{r^2-2r+2}\cdot c_2(E,h) = \frac{\omega^2\cdot r^2}{4\pi^2\cdot(r^2-2r+2)}\cdot\left(\lambda^2 - \frac{r-1}{r}\cdot\|F_-\|^2\right);$$

by assumption, the function $\psi = \lambda^2 - ((r-1)/r)\cdot\|F_-\|^2$ is strictly positive.

For a differentiable section s of E write with respect to local coordinates

$$\Phi_s := \frac{\sqrt{-1}}{2}\cdot h(F_h(s),s) = \frac{\sqrt{-1}}{2}\cdot\sum_{i,j=1}^{2}\phi_{ij}\,dz_i\wedge d\bar z_j.$$

The 2×2 matrix $\Phi = (\phi_{ij})$ is hermitian, so it is positive definite if and only if both $\operatorname{tr}(\Phi)$ and $\det(\Phi)$ are positive; this may be checked pointwise in suitable coordinates round a point $p\in X$. So choose normal coordinates such that

$$\omega(p) = \frac{\sqrt{-1}}{2}\cdot\sum_{i=1}^{2} dz_i\wedge d\bar z_i, \quad \omega^2(p) = -\tfrac{1}{2}\cdot dz_1\wedge d\bar z_1\wedge dz_2\wedge d\bar z_2.$$

The operator $\bigwedge$ adjoint to wedging with ω has the property

$$\textstyle\bigwedge F_- = 0, \ \bigwedge F_h = \bigwedge F_+ = -\sqrt{-1}\lambda\cdot id_E,$$

and in p one has $\bigwedge dz_i \wedge d\bar{z}_j = -\sqrt{-1}/2\delta_{ij}$ which implies

$$\operatorname{tr}(\Phi) = 4\bigwedge \Phi_s = 2\sqrt{-1}h\left(\bigwedge F_h(s), s\right) = 2\lambda|s|^2,$$

where $|s|^2 = h(s,s)$. Hence positivity of $\operatorname{tr}(\Phi)$ follows from the positivity of λ.

For the determinant, first observe that in p

$$\Phi_s \wedge \Phi_s = \det(\Phi)\cdot\omega^2;$$

the splitting $F_h = F_+ + F_-$ gives $\Phi_s = (\lambda/2)\cdot\omega\cdot|s|^2 + (\sqrt{-1}/2)\cdot h(F_-(s), s)$, hence

$$\Phi_s \wedge \Phi_s = \frac{\lambda^2}{4}\cdot|s|^4\cdot\omega^2 - \frac{1}{4}\cdot h(F_-(s), s)\wedge *\overline{h(F_-(s), s)}$$

$$= \frac{1}{4}(\lambda^2\cdot|s|^4 - \|h(F_-(s), s)\|_1^2)\cdot\omega^2$$

where $\|\ \|_1$ is the pointwise norm on 2-forms induced by g. Hence positivity of $\det(\Phi)$ follows by Proposition 2.3 below from $\operatorname{tr}(F_-) = 0$ and positivity of the function ψ.

b) In this case we unfortunately can only give a proof which makes extensive use of local coordinates; the somewhat tedious computations are left to the reader.

With respect to a local unitary frame field for E and normal coordinates in $p \in X$ the curvature F_h is given by a matrix $(F_{\alpha\beta})$ of $(1,1)$-forms

$$F_{\alpha\beta} = \sum_{i,j=1}^{n} F_{\alpha\beta ij}\, dz_i \wedge d\bar{z}_j;$$

because D_h is h-unitary one has

$$F_{\alpha\beta ij} = \bar{F}_{\beta\alpha ji}.$$

We define

$$a_{\alpha i} := F_{\alpha\alpha ii} - \frac{\lambda}{2};$$

then the assumption on $c_1(E,h)$ and the HE-condition imply the following relations:

$$\sum_\alpha F_{\alpha\alpha ij} = 0 \text{ for } i \neq j, \quad \sum_i F_{\alpha\beta ii} = 0 \text{ for } \alpha \neq \beta,$$

$$\sum_\alpha a_{\alpha i} = \sum_i a_{\alpha i} = 0,$$

$$\sum_\alpha F_{\alpha\alpha ii} = \frac{2}{n}\cdot\sum_i F_{\alpha\alpha ii} = \lambda.$$

We define the matrices $A = (A_{ij})$, $B = (B_{ij})$ by

$$A_{ij} = \begin{cases} F_{11ij} & \text{if } i \neq j \\ a_{1i} & \text{if } i = j \end{cases}, \quad B_{ij} = F_{12ij};$$

then A is hermitian and $\mathrm{tr}(A)=\mathrm{tr}(B)=0$. Using the relations above one calculates in p

$$\left(c_1^2(E,h)-\frac{4(n-1)^2}{n^2-2n+2}\cdot c_2(E,h)\right)\wedge\omega^{n-2}$$
$$=\frac{4}{\pi^2\cdot(n^2-2n+2)}\cdot\left(\frac{\lambda^2}{4}-\frac{n-1}{n}\cdot(|A|^2+|B|^2)\right)$$

where $|\ |$ is the usual matrix norm given by $|A|^2=\mathrm{tr}({}^tA\cdot\bar{A})$; by assumption we have

$$\frac{\lambda^2}{4}-\frac{n-1}{n}\cdot(|A|^2+|B|^2)>0.$$

The definition of positivity of F_h given above is equivalent to saying that the hermitian 2×2-matrix $K=(K_{\alpha\beta})$ given by

$$K_{\alpha\beta}=\sum_{i,j}x_iF_{\alpha\beta ij}\bar{x}_j$$

is positive definite for all $0\neq x=(x_1,\dots,x_n)\in\mathbb{C}^n$; again it suffices to look at trace and determinant. A simple calculation yields

$$\mathrm{tr}(K)=\frac{n\lambda}{2}\cdot|x|^2$$

which is positive because λ is, and

$$\det(K)=\frac{\lambda^2}{4}\cdot|x|^4-|{}^txA\bar{x}|^2-|{}^txB\bar{x}|^2;$$

this is positive by the inequality derived from the Chern forms and Lemma 2.2 below since $\mathrm{tr}(A)=\mathrm{tr}(B)=0$. ■

The real key of the proof given above is the following lemma which we haven't found in the literature; the simple proof has been provided by R. Brussee.

(2.2) LEMMA. *Let A be a complex $n\times n$-matrix such that* $\mathrm{tr}(A)=0$. *Then for all $x\in\mathbb{C}^n$ the following inequality holds*:

$$|{}^txA\bar{x}|^2\leq\frac{n-1}{n}\cdot|A|^2\cdot|x|^4.$$

PROOF.

$$|{}^txA\bar{x}|^2=|\mathrm{tr}(\bar{x}\,{}^txA)|^2$$
$$=\left|\mathrm{tr}\left(\left(\bar{x}\,{}^tx-\frac{|x|^2}{n}\cdot E_n\right)A\right)\right|^2$$

$$\leq \left| \bar{x}^t x - \frac{|x|^2}{n} \cdot E_n \right|^2 \cdot |A|^2 \text{ by Schwartz's inequality}$$

$$= \frac{n-1}{n} \cdot |x|^4 \cdot |A|^2. \qquad \blacksquare$$

As an application we get the proposition used in part a) of the proof; the different norms are written as there.

(2.3) PROPOSITION. *Let (E,h) be a hermitian vector bundle of rank r on a Kähler manifold (X,g) and F a $(1,1)$-form with values in* $\mathrm{End}(E)$ *such that* $\mathrm{tr}(F)=0$. *Then for every differentiable sections s of E one has*

$$\|h(F(s),s)\|_1^2 \leq \frac{r-1}{r} \cdot \|F\|^2 \cdot |s|^4.$$

PROOF. With respect to normal coordinates we write $F=\sum_{i,j} F_{ij} \cdot dz_i \wedge d\bar{z}_j$, then $\mathrm{tr}(F_{ij})=0$ for all i,j and

$$\|F\|^2 = \sum_{i,j} |F_{ij}|^2.$$

On the other hand,

$$\|h(F(s),s)\|_1^2 = \sum_{i,j} |h(F_{ij}(s),s)|^2,$$

so the claim follows from Lemma 2.2. $\blacksquare$

3. SOME REMARKS

As was said in the introduction, we can't verify the condition on the Chern forms in our criterion even for examples such as the Kodaira fibrations where the existence of a metric with positive curvature is known, so one might try to weaken this assumption. A first possibility would be to assume the positivity of the corresponding combination of Chern *classes* (i.e. the integrated version of the inequality holds); our argument then would work if one could show that the function ϕ appearing in Theorem 2.1 is constant. This is equivalent to saying that the second Chern form is harmonic, and we have no idea what sufficient conditions there are to guaranty that (surely it doesn't follow directly from the *HE*-equation).

Another try could be to look for a metric with positive curvature in the conformal class of the *HE*-metric h, i.e. to replace h by $h^g := e^g \cdot h$ with some function g. E.g. in the case $n=r=2$, the same method as above shows the following: If h is a *HE*-metric such that $c_1(E,h) \triangleq \omega$, then sufficient for positivity of the curvature F^g of h^g is that the following differential inequality for g holds:

$$(**) \qquad \triangle g - \|(\bar{\partial}\partial g) - \| > \frac{1}{\sqrt{2}} \cdot \|F_-\| - \lambda,$$

where $\triangle$ is the Laplace-Beltrami operator of g. Necessary for the existence of a solution of (**) is the negativity of the integral of the right hand side (so λ has to be positive) which would follow from $c_1^2(E) - 2c_2(E) > 0$; but again we don't know any sufficient condition for the solvability.

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